JOURNAL OF APPROXIMATION THEORY **82**, 434–439 (1995)

Integrating Completely Unisolvent Functions*

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Communicated by András Kroó

Received February 9, 1994; accepted in revised form September 7, 1994

We show that the integral of a completely n-unisolvent function defined on an interval is a completely $n+1$-unisolvent function. © 1995 Academic Press, Inc.

1. INTRODUCTION

In the following let f be a continuous function $\mathbb{R}^n \times I \to \mathbb{R}$ where $I \subset \mathbb{R}$ is some interval and let $INT(f)$ be the set of all functions $I \to \mathbb{R}: x \mapsto f(a_1, a_2, a_3, ..., a_n, x)$, $a_1, a_2, ..., a_n \in \mathbb{R}$. If $n \geqslant 2$, let f_a, $a \in \mathbb{R}$ be the map $\mathbb{R}^{n-1} \times I \to \mathbb{R}: (a_2, a_3, ..., a_n, x) \mapsto f(a, a_2, a_3, ..., a_n, x)$. A function $g \in INT(f)$ *interpolates* $m \in \mathbb{N}$ points $(x_i, y_i) \in I \times \mathbb{R}$, $i \in \{1, 2, ..., m\}$ if $g(x_i) = y_i$ for all $i \in \{1, 2, ..., m\}$.

We say that f is *n-unisolvent* (or unisolvent of degree n) if for any choice of n points $(x_i, y_i) \in I \times \mathbb{R}$, $i \in \{1, 2, ..., n\}$, $x_1 < x_2 < \cdots < x_n$ there exists a uniquely determined $g \in INT(f)$ that interpolates all n points. The classical example for such a function is $p_n: \mathbb{R}^n \times I: (a_1, a_2, ..., a_n, x) \mapsto \sum_{k=1}^n a_k x^{n-k}$.

If $n = 1$, we will say that f is *completely* 1-*unisolvent* if it is 1-unisolvent. For $n \geqslant 2$ we say that f is *completely n-unisolvent* if and only if:

(1) the function f is n-unisolvent;

(2) for all $a \in \mathbb{R}$ the function f_a is completely $(n-1)$-unisolvent.

Let $\mathcal{I}_n^I(\bar{\mathcal{I}}_n^I)$ be the set of all (completely) n-unisolvent functions $\mathbb{R}^n \times I \to \mathbb{R}$. The function $p_n, n \in \mathbb{N}$, as we defined it above, is contained in $\bar{\mathcal{I}}_n^I$. Note that this is just the interpolation system $\{1, x, x^2, ..., x^{n-1}\}$ combined into a completely n-unisolvent function. More generally, if $\{u_i\}_{i=0}^{n-1}$ is a complete Chebyshev system (see, e.g., [1]) of continuous functions defined on the interval I, then $\mathbb{R}^n \times I \to \mathbb{R}: (a_1, a_2, ..., a_n, x) \mapsto \sum_{k=1}^n a_k u_{n-k}$ is a completely n-unisolvent function.

Here are some examples of completely 2-unisolvent functions that do not arise from complete Chebyshev systems in this manner: Let $h: \mathbb{R} \to \mathbb{R}$ be a

* This research was supported by a Feodor Lynen fellowship.

0021-9045/95 $12.00

continuously differentiable function with a bijective derivative. Then f^h: $\mathbb{R}^2 \times \mathbb{R}$: $(a_1, a_2, x) \mapsto h(x + a_1) + a_2$ is a completely 2-unisolvent function. Examples for h are the functions $\mathbb{R} \to \mathbb{R}$: $x \mapsto x^{2n}$, $n \in \mathbb{N}$. In [5, §2] it is shown that the graphs of the functions in $INT(f^h)$ together with the verticals in the xy-plane form the line set of an affine plane that has the xy-plane as its point set. The sets $INT(f_a^h)$, $a \in \mathbb{R}$ correspond to parallel classes of lines in this plane. Affine planes like this are examples of 2-dimensional affine planes. In [3] we show that any 2-dimensional affine plane corresponds to a completely 2-unisolvent function. In the same paper we also show how (in general non-linear) completely 3-unisolvent functions can be constructed from so-called 2-dimensional Laguerre planes.

Given an arbitrary (completely) n-unisolvent function $f \in \mathscr{I}_n^I (f \in \bar{\mathscr{I}}_n^I)$ and a subinterval I' of I, it is clear that the "restriction" of f to I' is also a (completely) n-unisolvent function.

For more information about unisolvent functions the reader is referred to [4] and [6].

2. Integrating Completely Unisolvent Functions

Let $I \subset \mathbb{R}$ be an interval that contains the point b. For every $f \in \mathscr{I}_n^I$ we define

$$S^b(f)\colon \mathbb{R}^{n+1} \times I \to \mathbb{R}\colon (a_1, a_2, ..., a_{n+1}, x) \mapsto \int_b^x f(a_1, a_2, ..., a_n, t)\, dt + a_{n+1}.$$

The aim of this note is to show that

Proposition 2.1. *Let $f \in \bar{\mathscr{I}}_n^I$, $n \in \mathbb{N}$ where I is an interval that contains the point b. Then $S^b(f) \in \bar{\mathscr{I}}_{n+1}^I$.*

In the special case where f arises from a complete Chebyshev system, as described above, this result is well-known (see, e.g., [7, Lemma 13.2]).

In order to be able to prove 2.1 we will make constant use of some facts we want to fix in the form of two lemmas. As a special case of what Tornheim proved in [6, Theorem 5] we have

Lemma 2.2. *Let $I \subset \mathbb{R}$ be a closed interval, let $f \in \mathscr{I}_n^I$, and let n sequences of points $\{(x_{i,j}, y_{i,j})\}_{i \in \mathbb{N}}$, $j \in \{1, 2, ..., n\}$ in $I \times \mathbb{R}$ converge to n points (x_j, y_j), respectively, such that $x_{i,j} \neq x_{i,k}$ and $x_j \neq x_k$ if $j \neq k$. Furthermore, for all $i \in \mathbb{N}$ let f^i be the uniquely determined function in $INT(f)$ that interpolates the n points $(x_{i,j}, y_{i,j})$, $j \in \{1, 2, ..., n\}$. Then the sequence of functions $\{f^i\}_{i \in \mathbb{N}}$ converges uniformly to the uniquely determined function in $INT(f)$ that interpolates the n points (x_j, y_j), $j \in \{1, 2, ..., n\}$.*

We also need the following

LEMMA 2.3. *Let $I \subset \mathbb{R}$ be some interval that contains* 0, *let $f \in \mathcal{I}_1^I$ and $f(1,0) > f(0,0)(f(1,0) < f(0,0))$. Then all functions $\mathbb{R} \to \mathbb{R}$: $a \mapsto f(a,x)$, $x \in I$ are strictly increasing (decreasing) homeomorphisms. Furthermore, given $x_1, x_2 \in I$, $x_1 < x_2$, the function g: $\mathbb{R} \to \mathbb{R}$: $a \mapsto \int_{x_1}^{x_2} f(a,t)\,dt$ is a strictly increasing (decreasing) homeomorphism.*

Proof. Let $f(1,0) > f(0,0)$. Let $x \in I$. If $f(1,x) \leqslant f(0,x)$, then there exists an $x^* \in I$ such that $f(1,x^*) = f(0,x^*)$. This is impossible since $f \in \mathcal{I}_1^I$. The function $\mathbb{R} \to \mathbb{R}$: $a \mapsto f(a,x)$ is continuous and bijective by definition, i.e., it is a homeomorphism. Since $f(1,x) > f(0,x)$, this homeomorphism is strictly increasing. Now it is clear that the function g is continuous and strictly increasing. We show that g is bijective. For all $a \in \mathbb{R}$ let $m_a := \min_{x \in [x_1, x_2]} \{f(a,x)\}$ and let x_a be a point in $[x_1, x_2]$ where f_a assumes this minimum. The function $\mathbb{R} \to \mathbb{R}$: $a \mapsto m_a$ is clearly strictly increasing and it therefore suffices to show that $\lim_{a \to \infty} m_a = \infty$ to make sure that indeed $\lim_{a \to \infty} \int_{x_1}^{x_2} f(a,t)\,dt = \infty$. Let $\{a_i\}_{i \in \mathbb{N}}$ be a strictly increasing sequence of real numbers such that $\lim_{i \to \infty} a_i = \infty$. Assume the sequence $\{m_{a_i}\}_{i \in \mathbb{N}}$ has a finite accumulation point m^*, i.e., $\lim_{i \to \infty} m_{a_i} = m^*$. Then we can find a subsequence $\{a_i'\}_{i \in \mathbb{N}}$ of $\{a_i\}_{i \in \mathbb{N}}$ and an $x^* \in [x_1, x_2]$ such that $\lim_{i \to \infty} m_{a_i'} = m^*$ and $\lim_{i \to \infty} x_{a_i'} = x^*$. Since f is 1-unisolvent there is a uniquely determined $a^* \in \mathbb{R}$ such that $f_{a^*}(x^*) = m^*$. By Lemma 2.2, we know that the sequence of functions $\{f_{a_i'}\}_{i \in \mathbb{N}}$ converges uniformly to the function f_{a^*} on the interval $[x_1, x_2]$. This implies that $m^* = \min_{x \in [x_1, x_2]} \{f(a^*, x)\}$. Let $i \in \mathbb{N}$ be such that $a_i' > a^*$. Then $m^* < m_{a_i'}$. This is a contradiction. We can use a similar argument to show that $\lim_{a \to -\infty} \int_{x_1}^{x_2} f(a,t)\,dt = -\infty$. Hence g is bijective.

The respective conclusions in the case $f(1,0) < f(0,0)$ can be derived in a similar fashion. ∎

Proof of 2.1. W.l.o.g. we may assume that $b = 0$. We abbreviate $S^b(f)$ by $S(f)$.

We are going to use induction on n to prove this result. Let $n = 1$. Furthermore, let $a \in \mathbb{R}$. Then $S(f)_a$ is the function $\mathbb{R} \times I \to \mathbb{R}$: $(a_2, x) \mapsto \int_0^x f(a,t)\,dt + a_2$. This function is clearly completely 1-unisolvent. Let $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$, $x_0, x_1 \in I$, $x_0 < x_1$. We have to show that there is a uniquely determined function in $INT(S(f))$ that interpolates both points. The functions that interpolate the point (x_0, y_0) are the functions $I \to \mathbb{R}$: $x \mapsto y_0 + \int_{x_0}^x f(a,t)\,dt$, $a \in \mathbb{R}$. It therefore suffices to show that the function $\mathbb{R} \to \mathbb{R}$: $a \mapsto \int_{x_0}^{x_1} f(a,t)\,dt$ is bijective. By Lemma 2.3, this is the case.

Let the statement in the proposition be true for all functions in $\mathcal{I}_{n-1}^I$ for some $n \geqslant 2$ and let $f \in \mathcal{I}_n^I$. Furthermore, let $a \in \mathbb{R}$. Then $S(f)_a = S(f_a)$.

Since $f_a \in \mathscr{I}^I_{n-1}$ we conclude that $S(f)_a \in \mathscr{I}^I_n$. Now we have to show the following: Let (x_0, y_0), (x_1, y_1), ..., $(x_{n-1}, y_{n-1}) \in I \times \mathbb{R}$, $x_i < x_{i+1}$, $i \in \{0, 1, ..., n-2\}$ be n points, and let $x_n \in I$, $x_{n-1} < x_n$. Then for all $y \in \mathbb{R}$ there exists a uniquely determined function in $INT(S(f))$ that interpolates the n points (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}) and the point (x_n, y). The functions that interpolate the (first) n points are the functions $I \to \mathbb{R}$: $x \mapsto y_0 + \int_{x_0}^{x} f(a, \phi_1(a), \phi_2(a), ..., \phi_{n-1}(a), t)\, dt$, $a \in \mathbb{R}$ where the functions ϕ_i: $\mathbb{R} \to \mathbb{R}$, $i \in \{1, 2, ..., n-1\}$ are uniquely determined by the $n-1$ equations

$$\int_{x_0}^{x_i} f(a, \phi_1(a), \phi_2(a), ..., \phi_{n-1}(a), t)\, dt = y_i - y_0,\ i = 1, 2, ..., n-1$$

since for all $a \in \mathbb{R}$ the function $S(f)_a$ is n-unisolvent. This system of equations is equivalent to the following system of equations

$$\int_{x_{i-1}}^{x_i} f(a, \phi_1(a), \phi_2(a), ..., \phi_{n-1}(a), t)\, dt = y_i - y_{i-1},\ i = 1, 2, ..., n-1.$$

Let f^a: $I \to \mathbb{R}$: $x \mapsto f(a, \phi_1(a), \phi_2(a), ..., \phi_{n-1}(a), x)$. So the functions in $INT(S(f))$ that interpolate the first n points are the functions $I \to \mathbb{R}$: $x \mapsto y_0 + \int_{x_0}^{x} f^a(t)\, dt$, $a \in \mathbb{R}$.

We show that the functions ϕ_i are continuous. Let J be the open interval (x_0, x_n) and let $D := \{(x'_1, x'_2, ..., x'_{n-1}, a'_1, a'_2, ..., a'_n) \mid x'_i \in J, x'_1 < x'_2 < \cdots < x'_{n-1}, a'_i \in \mathbb{R}\}$. Then $D \subset \mathbb{R}^{2n-1}$ and the function

$$\begin{aligned} g\colon D \to D\colon & (x'_1, x'_2, ..., x'_{n-1}, a'_1, a'_2, ..., a'_n) \\ & \mapsto \left(x'_1, ..., x'_{n-1}, a'_1, \int_{x_0}^{x'_1} f(a'_1, a'_2, ..., a'_n, t)\, dt, \right. \\ & \left. \int_{x'_1}^{x'_2} f(a'_1, a'_2, ..., a'_n, t)\, dt, ..., \int_{x'_{n-2}}^{x'_{n-1}} f(a'_1, a'_2, ..., a'_n, t)\, dt \right) \end{aligned}$$

is continuous and bijective. Since D is an open subset of $\mathbb{R}^{2n-1}$, g is a homeomorphism by "Brouwer's theorem on the invariance of domain" (see, e.g., [2]) which guarantees that a continuous bijection of a manifold is a homeomorphism. We conclude that the $n+i$-th component $\bar{\phi}_i$: $D \to \mathbb{R}$ of the continuous functions g^{-1} is itself continuous and that for all $a \in \mathbb{R}$ we have $\phi_i(a) = \bar{\phi}_i(x_1, x_2, ..., x_{n-1}, a, y_1 - y_0, y_2 - y_1, ..., y_{n-1} - y_{n-2})$. This implies that the functions ϕ_i are continuous which in turn guarantees that h: $\mathbb{R} \to \mathbb{R}$: $a \to \int_{x_0}^{x_n} f^a(t)\, dt$ is a continuous function. We show that h is injective. Let $a_1, a_2 \in \mathbb{R}$, $a_1 \neq a_2$. Then $\int_{x_i}^{x_{i+1}} (f^{a_1}(t) - f^{a_2}(t))\, dt = 0$ for all $i \in \{0, 1, ..., n-2\}$, which implies that there exist $x'_i \in]x_i, x_{i+1}[$, $i \in \{0, 1, ..., n-2\}$ such that $f^{a_1}(x'_i) = f^{a_2}(x'_i)$. Since f is n-unisolvent these are the only such values in the whole of I. This means that everywhere in the interval $]x_{n-1}, x_n[$ we have $f^{a_1}(x) < f^{a_2}(x)$ or $f^{a_1}(x) > f^{a_2}(x)$. Hence $\int_{x_0}^{x_n} f^{a_1}(t)\,dt \neq \int_{x_0}^{x_n} f^{a_2}(t)\,dt$. Of course, this just means that the function h is injective. Hence its image has to be an open interval. It remains to show that this interval is all of $\mathbb{R}$, i.e., that h is bijective.

W.l.o.g., let us assume that h is a strictly increasing function. So what we have to verify is that $\lim_{a \to \pm\infty} h(a) = \pm\infty$. Since h is strictly increasing, for all $x \in]x_{n-1}, x_n]$ the function $g'_x: \mathbb{R} \to \mathbb{R}: a \to f^a(x)$ is also strictly increasing. As a consequence of this we know that the limits $\lim_{a \to \pm\infty} h(a)$ and $\lim_{a \to \pm\infty} g'_x(a)$, $x \in]x_{n-1}, x_n]$ exist (finite or infinite). Let us assume that $\lim_{a \to \infty} h(a) < \infty$, or equivalently, that $\lim_{a \to \infty} \int_{x_{n-1}}^{x_n} f^a(t)\,dt < \infty$. Then we can find n distinct values $x_i^* \in]x_{n-1}, x_n]$, $i \in \{0, 1, \ldots, n-1\}$ such that $\lim_{a \to \infty} g'_{x_i^*}(a) = y_i^* < \infty$. (If this were not possible we would be able to find a subinterval $[x'_{n-1}, x'_n]$ of $[x_{n-1}, x_n[$ such that the function

$$\mathbb{R} \times [x'_{n-1}, x'_n] \to \mathbb{R}: (a, x) \mapsto \begin{cases} f^a(x) & \text{for } a > 0 \\ f^0(x) + a & \text{for } a \leqslant 0 \end{cases}$$

is 1-unisolvent. The existence of such a subinterval would already guarantee, by Lemma 2.3, that $\infty = \lim_{a \to \infty} \int_{x'_{n-1}}^{x'_n} f^a(t)\,dt \leqslant \int_{x_{n-1}}^{x_n} f^a(t)\,dt$, which is a contradiction to our assumption.) Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of real numbers such that $\lim_{k \to \infty} a_k = \infty$. Now f^{a_k}, $k \in \mathbb{N}$ is the uniquely determined function in $INT(f)$ that interpolates the n points $(x_i^*, f^{a_k}(x_i^*))$, $i \in \{0, 1, \ldots, n-1\}$. As k goes to infinity these points tend towards the n points (x_i^*, y_i^*), respectively. Let f^* be the uniquely determined function in $INT(f)$ that interpolates these points. Now Lemma 2.2 guarantees that f^* is the uniform limit of the sequence of functions $\{f^{a_k}\}_{k \in \mathbb{N}}$. This implies that for all $i \in \{0, 1, \ldots, n-2\}$ we find $\int_{x_i}^{x_{i+1}} f^*(t)\,dt = y_{i+1} - y_i$, i.e., there has to exist an $a^* \in \mathbb{R}$ such that $f^* = f^{a^*}$. Let $k \in \mathbb{N}$ be such that $a_k > a^*$. Then $f^{a_k}(x_i^*) > y_i^*$. This is a contradiction. We conclude that $\lim_{a \to \infty} h(a) = \infty$.

A similar argument shows that $\lim_{a \to -\infty} h(a) = -\infty$. This completes the proof of the proposition. ▮

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